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Scattering of a Klein–Gordon Particle by a Black Hole

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By using the curved space-time Klein–Gordon equation, the form of the wave function of a scalar particle near a nonrotating black hole is obtained. It is shown that although the radial wave function oscillates infinitely rapidly near the black hole, the probability density remains finite even on the event horizon. This is consistent with the fact that the Schwarzschild surface is nonsingular. An expression is given for the large angular momentum scattering differential cross section by comparing the asymptotic form of the radial wave equation with the corresponding Coulomb radial wave equation in ordinary quantum mechanics.

1. INTRODUCTION

A problem which has received considerable attention in general relativity is the exact nature of the surface $r = 2M$ in the Schwarzschild metric for the empty region surrounding an isolated body of mass M located at the spatial origin. A thorough understanding of this apparently singular surface is essential in the study of a phenomenon such as gravitational collapse. Although the above singularity can be removed by means of an analytic continuation (Fronsdal, 1959) or a coordinate transformation (Kruskal, 1960), there have been several attempts to establish physically that the Schwarzschild surface is indeed nonsingular. For example, the invariants associated with the curvature tensor are found to be nonsingular at $r = 2M$. The nonsingular nature of the surface $r = 2M$ in the space-time manifold has also been established by analyzing the light cone along a radial geodesic (Finkelstein, 1958; Misner, 1968). It is now generally accepted that the

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Schwarzschild surface forms only an absolute event horizon (Penrose, 1969). More recently, it was shown (Prasanna, 1972) that the relative acceleration between neighboring particles in the field of an isolated mass M is finite at the surface $r = 2M$ and hence the surface is nonsingular on purely physical grounds.

In this paper, we investigate the behavior of the wave function and the corresponding probability density of a scalar particle of mass μ near a superdense body of mass M which has a geometrical radius $r \sim 2M$. This might be the case for an object that is collapsing beyond the event horizon to form a black hole. We also consider the scattering of the scalar particle by such a black hole.

The plan of the paper is as follows. In Section 2, we derive the appropriate radial Klein–Gordon equation for a scalar particle in the curved space-time background of the exterior Schwarzschild metric. In Section 3, we derive the solutions of the radial wave equation valid near the Schwarzschild event horizon $r = 2M$ and examine the (radial) probability density. In Section 4, we give an expression for the scattering differential cross section by comparison with the usual nonrelativistic Coulomb scattering problem. Section 5 contains some conclusions.

2. THE CURVED SPACE-TIME KLEIN–GORDON WAVE EQUATION

The curved space-time Klein–Gordon wave equation for a scalar particle of rest mass $\mu\hbar/c$ is given by

$$(\Box + \mu^2)\Psi = 0 \tag{1}$$

in units where $\hbar = c = 1$, where $\Box$ denotes the curved space-time wave operator defined by

$$\Box \equiv g^{ik}\nabla_i\nabla_k \tag{2}$$

where ∇_i denotes covariant derivative and Ψ is the wave function of the particle.

Now, for a static spherically symmetric body of mass M, we have for the exterior field the usual Schwarzschild solution

$$\begin{aligned} ds^2 &= (1-2M/r)\,dt^2 - (1-2M/r)^{-1}\,dr^2 - r^2\,d\Omega^2 \\ d\Omega^2 &\equiv d\theta^2 + \sin^2\theta\,d\phi^2 \end{aligned} \tag{3}$$

in spherical polar coordinates (t, r, θ, ϕ).

Using the metric (3), we find by direct calculation that the wave equation (1) takes the form

$$\begin{aligned}\big[&(1-2M/r)^{-1}\,\partial^2/\partial t^2-(1-2M/r)\,\partial^2/\partial r^2-(2/r)(1-M/r)\,\partial/\partial r\\&-(1/r^2)\,\partial^2/\partial\theta^2-(1/r^2)\cot\theta\,\partial/\partial\theta\\&-(1/r^2\sin^2\theta)\,\partial^2/\partial\phi^2+\mu^2\big]\Psi(t,r,\theta,\phi)=0\end{aligned}\tag{4}$$

Assuming that the variables can be separated in the usual way, we put

$$\Psi(t,r,\theta,\phi)=T(t)R(r)\Theta(\theta,\phi)\tag{5}$$

in equation (4) and obtain the following three equations (6)–(8):

$$\frac{d^2T}{dt^2}+Tw^2=0\tag{6}$$

where w is a separation constant, corresponding to the frequency of the wave;

$$\begin{aligned}&(1-2M/r)^2\,d^2R/dr^2+(2/r)(1-2M/r)(1-M/r)\,dR/dr\\&\quad+\big[w^2-(1-2M/r)\mu^2-(1-2M/r)l(l+1)/r^2\big]R=0\end{aligned}\tag{7}$$

where l is a nonnegative integer, which is the orbital angular momentum quantum number of the scalar particle;

$$\big[\partial^2/\partial\theta^2+\cot\theta\,\partial/\partial\theta+\mathrm{cosec}^2\theta\,\partial^2/\partial\phi^2+l(l+1)\big]\Theta=0\tag{8}$$

Equation (8) has the usual solution

$$\Theta_{lm}(\theta,\phi)=Y_l^m(\cos\theta)\exp(im\phi)\tag{9}$$

where $Y_l^m(\cos\theta)$ are spherical harmonics and m is the magnetic quantum number, an integer such that $|m|\leqslant l$, while equation (6) has the general solution

$$T(t)=Ae^{-iwt}+Be^{iwt}\tag{10}$$

where A, B are arbitrary constants. Hence the eigensolutions of the scalar wave equation (4) are given by

$$\Psi(t,r,\theta,\phi)=NR(r)Y_l^m(\cos\theta)\exp i(m\phi\mp wt)\tag{11}$$

where $R(r)$ is a solution of the radial wave equation (7) and N is a normalization factor.

Now, the radial wave equation (7) is equivalent to

$$(1/r^2)(1-2M/r)(d/dr)\left[r^2(1-2M/r)\,dR/dr\right] + \left[w^2-(1-2M/r)\mu^2-(1-2M/r)l(l+1)/r^2\right] R = 0 \quad (12)$$

and, on putting

$$R = u/r$$

this radial equation takes the form

$$(1-2M/r)(d/dr)\left[(1-2M/r)\,du/dr\right] + \left[w^2-(1-2M/r)\left\{\mu^2+2M/r^3+l(l+1)/r^2\right\}\right]u = 0 \quad (13)$$

Finally, the substitution

$$dr^* = dr(1-2M/r)^{-1}$$
$$r^* = r+2M\ln(r/2M-1) \quad (14)$$

reduces (13) to the effective-potential form

$$d^2u/dr^{*2} + \left[w^2-(1-2M/r)\left\{\mu^2+2M/r^3+l(l+1)/r^2\right\}\right]u = 0 \quad (15)$$

where

$$r = r(r^*)$$

which has the form of a one-dimensional wave equation with an independent variable r^*. The effective potential $V(r^*)$ is given by

$$V^2(r^*) = (1-2M/r)\left\{\mu^2+2M/r^3+l(l+1)/r^2\right\}, \qquad r = r(r^*) \quad (16)$$

which clearly vanishes on the event horizon $r = 2M$.

3. RADIAL SOLUTION NEAR EVENT HORIZON

In the region near the surface $r = 2M$, let

$$r = 2M + x$$

where $x \ll 2M$. Then we find that the radial wave equation (7) reduces to

$$\left[d^2/dx^2 + (1/x - 1/2M)\,d/dx + \left(a + b/x + p/x^2\right)\right] y(x) = 0 \quad (17)$$

on neglecting $(x/2M)^3$ and higher orders, where a, b, p are constants given, respectively, by

$$a = \mu^2 + 3l(l+1)/(4M^2), \qquad b = -\left[2M\mu^2 + l(l+1)/(2M)\right]$$

$$p = 4M^2w^2 \tag{18}$$

Equation (17) is a confluent hypergeometric differential equation which on integration leads in the general case to a confluent hypergeometric function and in a particular case to a Bessel function (Bateman, 1953). In the general case when

$$q = 16M^2\mu^2 + 12l(l+1) \neq 1 \tag{19}$$

the solution of (17) is

$$y(x) = x^{-1/2}\exp(x/4M)W(\tau, \zeta, \xi) \tag{20}$$

where

$$\tau = \left[\{1 - 2l(l+1) - 8M^2\mu^2\}(1-q)^{1/2}\right]/(8M^2)$$

$$\zeta = \pm 2iMw, \qquad \xi = (x/2M)(1-q)^{1/2}$$

and $W(\tau, \zeta, x)$ is Whittaker's function which is a solution of the differential equation

$$d^2z/dx^2 + \left\{-1/4 + \tau/x + (1/4 - \zeta^2)/x^2\right\}z = 0 \tag{21}$$

The Whittaker function $W(\tau, \zeta, x)$ is given in terms of the confluent hypergeometric function ${}_1F_1(p; s; x)$ by

$$W(\tau, \zeta, x) = x^{1/2+\zeta}\exp(-x/2)\,{}_1F_1(1/2 + \zeta - \tau; 1 + 2\zeta; x) \tag{22}$$

where ${}_1F_1(p; s; x) \equiv \sum_{n=0}^{\infty}[(p)_n x^n/(s)_n n!]$, in the usual notation. On the other hand, if $q = 1$, the solution of (17) is

$$y(x) = \exp(x/4M)J_\nu(\xi) \tag{23}$$

where $\nu = \pm 2iMw$, $\xi = [\{1 - 2l(l+1) - 8M^2\mu^2\}(x/M)]^{1/2}$, and $J_\nu(x)$ is a solution of the Bessel equation

$$x^2d^2y/dx^2 + x\,dy/dx + (x^2 - \nu^2)y = 0 \tag{24}$$

Now, in the region near the event horizon, we have $x \sim 0$ and so $\xi \sim 0$. Hence using (22) in (20) we obtain for the general case the asymptotic form

$$y(x) \sim Cx^{\xi}(1 + x/4M) \tag{25}$$

where the complex amplitude C is given by

$$C = \left[(1-q)/(4M^2)\right]^{1/4+\xi/2}$$

Combining this solution with the angular part (9) and the solution (10), we obtain for the form of the wave function near the Schwarzschild surface

$$\Psi \sim Cx^{\xi}(1 + x/4M)Y_l^m(\cos\theta)\exp i(m \mp wt) \tag{26}$$

The radial function $x^{\xi} = \exp(\pm 2iMw \ln x)$ gives a rapid oscillatory character to the ingoing and outgoing waves as $x \to 0_+$ (i.e., near the Schwarzschild event horizon). However, if we use the boundary condition in Matzner (1968), namely, that at $r = 2M$, the waves are pure ingoing (i.e., there is no scalar radiation from the black hole), the (radial) probability density at the event horizon is given by (25) as

$$P_r = \left[|R(r)|^2\right] = \frac{r}{2M} + \left(1 - \frac{r}{2M}\right)^2 \tag{27}$$

and so is finite and tends to 1 as $r \to 2M_+$.

The corresponding (radial) probability current density (flux vector) $\mathbf{S}(\mathbf{r},t) = \mathscr{R}(\overline{\Psi}(h/im)\nabla\Psi)$ is also found to be

$$S_r = \frac{2w}{r-2M}\left[\frac{r}{2M} + \left(1 - \frac{r}{2M}\right)^2\right] \tag{28}$$

which $\to \infty$ as $r \to 2M_+$.

4. SCATTERING OF THE SCALAR PARTICLE

We now derive the asymptotic form of the radial wave equation (7) and compare the result with the corresponding nonrelativistic Coulomb problem as was done in Matzner (1968) in the case of a massless scalar particle. Instead of the coordinate transformation (14) which contains a logarithmic term and hence is not suitable for the asymptotic region, one can use the transformation

$$R = \frac{1}{r(1-2M/r)^{1/2}}u(r)$$

to bring equation (7) into the form

$$\frac{d^2u}{dr^2}+\left[w^2/(1-2M/r)^2-\mu^2/(1-2M/r)-l(l+1)/\{r^2(1-2M/r)\}\right.$$
$$\left.+M^2/\{r^4(1-2M/r)^2\}\right]u(r)=0 \tag{29}$$

It has not been possible to solve this radial equation in a closed form. However, in the asymptotic region, $r\to\infty$ and $r \gg 2M$. Hence (29) gives the asymptotic form of the radial equation as

$$\frac{d^2u}{dr^2}+\left[w^2-\mu^2+(4Mw^2-2M\mu^2)/r\right.$$
$$\left.+\{12M^2w^2-4M^2\mu^2-l(l+1)\}/r^2\right]u(r)=0 \tag{30}$$

on neglecting $(2M/r)^3$ and higher orders. Here the term in $1/r$ is an attractive Newtonian coupling for $2w^2>\mu^2$.

On redefining a modified orbital angular momentum quantum number l' by

$$l'(l'+1)=l(l+1)-12M^2w^2+4M^2\mu^2$$

and dropping the dash, (30) takes the form

$$\frac{d^2u}{dr^2}+\left[w^2-\mu^2+(4Mw^2-2M\mu^2)/r-l(l+1)/r^2\right]u(r)=0 \tag{31}$$

which we can compare with the Coulomb radial equation

$$\frac{d^2y_l}{dr^2}+\left[k^2-2\gamma k/r-l(l+1)/r^2\right]y_l=0 \tag{32}$$

to find the correspondence

$$\gamma\langle\leftrightarrow\rangle(2Mw^2-M\mu^2)/(w^2-\mu^2)^{1/2}$$

and

$$k^2\langle\leftrightarrow\rangle w^2-\mu^2$$

The regular partial wave solution of (31) therefore has the asymptotic form

$$R = \frac{1}{r} u(r) r \xrightarrow{} \infty \frac{1}{r} \sin\left(kr - \gamma \ln 2kr - \frac{1}{2} l\pi - \sigma_l \right) \tag{33}$$

where $-\sigma_l$ is the equivalent Coulomb phase shift given by

$$\sigma_l = \arg \Gamma(l + 1 + i\gamma) \tag{34}$$

where

$$\gamma = (M\mu^2 - 2Mw^2)/(w^2 - \mu^2)^{1/2} \tag{35}$$

On ignoring the logarithmic term in (33), the scattering differential cross section for the equivalent Coulomb problem is then given, for large l, by

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2k)^2} \left| \Sigma(2l+1)\exp(-2i\sigma_l) P_l(\cos\theta) \right|^2 \tag{36}$$

for $\theta \neq 0$, where $k^2 = w^2 - \mu^2$, and the forward-scattering amplitude is infinite. This result is formally identical with that for a massless scalar particle in Matzner (1968), except that the frequency is now modified according to $w^2 \to (w^2 - \mu^2)$.

5. CONCLUSIONS

We have shown that although the wave function of a scalar particle oscillates infinitely rapidly in the neighbourhood of a black hole of mass M, the probability density remains finite there. It is well known that the wave function Ψ is, per se, not a physical entity, whereas the probability density $|\Psi|^2$ is. The above investigation therefore confirms physically that the Schwarzschild surface is indeed nonsingular.

Finally, the particle is scattered and for large l, the equivalent Coulomb phase shift and differential cross section are given by (34) and (36), respectively.

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